Toward Optimal Radio Colorings of Hypercubes via SAT-Solving

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ATS Science Reading Group
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Summary

We used SAT-solving for improving the best bounds on a math problem, providing insights into:

- Cube and Conquer
- Symmetry Breaking (both internal and external)
- *Solver-guidance.*

The two kinds of symmetry breaking

Consider the problem of finding integers a and b such that $a + b = 100$.

- If (a, b) is a solution, then so is (b, a).
- If (a, b) is a solution, then so is $(a + 5, b - 5)$.
- Can we assume there will be a solution with $a = b$?

The two kinds of symmetry breaking

Consider the problem of finding integers a and b such that $a + b = 100$.

- If (a, b) is a solution, then so is (b, a). (Symmetry between solutions)
- If (a, b) is a solution, then so is $(a + 5, b - 5)$.
  (Symmetry between solutions)
- Can we assume there will be a solution with $a = b$?
  (Symmetry within solutions)

40 Years of Successes in Computer-Aided Mathematics

1976 Four-Color Theorem

1998 Kepler Conjecture

2010 "God's Number = 20": Optimal Rubik's cube strategy

2012 At least 17 clues for a solvable Sudoku puzzle

2014 Boolean Erdős discrepancy problem

2016 Boolean Pythagorean triples problem

2018 Schur Number Five

2019 Keller's Conjecture

2022 Packing Number of Square Grid

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Definition (Radio 2-coloring)

Given a graph $G = (V, E)$, and a natural number $s \geq 1$, a radio 2-coloring of G of span s is a function $f : V(G) \rightarrow \{0, \ldots, s\}$ such that:

1. $f(u) \neq f(v)$, if u and v are at distance 2.
2. $|f(u) - f(v)| \geq 2$, if u and v are at distance 1.

Definition (Radio 2-chromatic number)

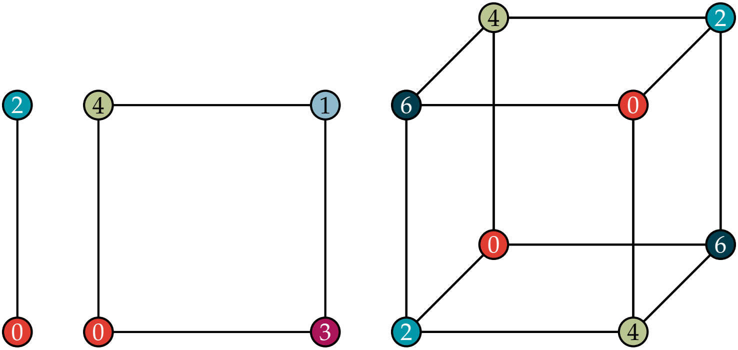
The radio 2-chromatic number of a graph G, denoted by $\lambda(G)$, is the smallest s for which G admits a radio 2-coloring with span s.

Radio 2-Coloring of Hypercubes

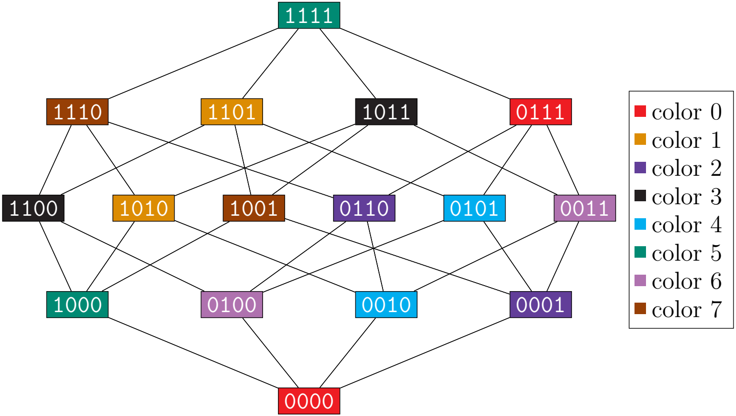
Definition

For any natural number $n \geq 1$, the hypercube graph of order n, denoted by Q_n, has vertex set $\{0,1\}^n$, and edges between elements of $\{0,1\}^n$ that differ in exactly one component.

Example



Internal Symmetry of Optimal Q_4 Coloring



An internal symmetry is a mapping of a coloring onto itself

- Interested in non-trivial mappings
- Mapping onto itself by applying XOR mask 0111

Coding Theory Results

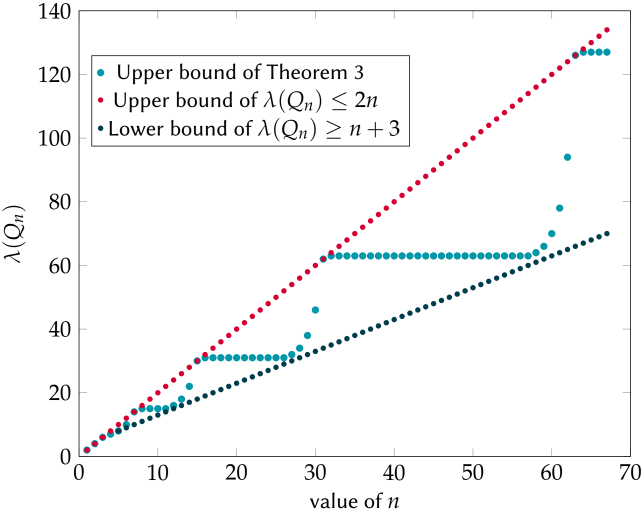
Theorem (1, Griggs and Yeh (1992), Jonas (1993))
For any $n \geq 5$, *we have* $n + 3 \leq \lambda(Q_n) \leq 2n + 1$.

Theorem (2, Whittlesey, Georges, and Mauro (1995))
For any $n \geq 1$, *we have* $\lambda(Q_n) \leq 2n$.

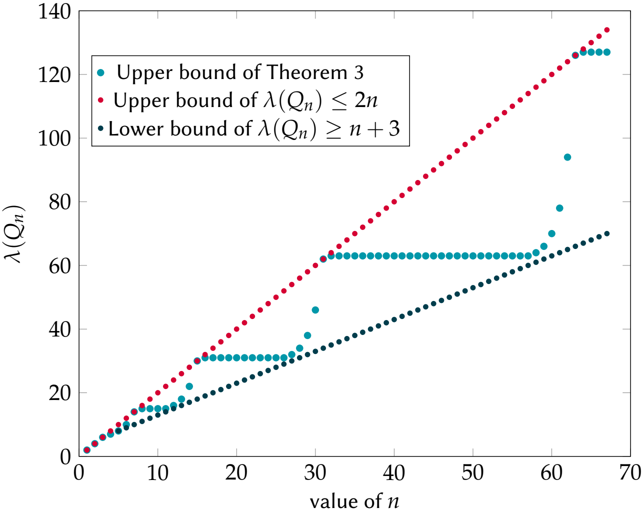
Theorem (3, Whittlesey, Georges, and Mauro (1995))
For any $k \geq 1$ *and* $q \leq k + 1$, *we have*

$$\lambda(Q_{2^k - q}) \leq 2^k + 2^{k-q+1} - 2.$$

Upper and Lower Bounds



Upper and Lower Bounds



What is the behavior in the limit?

Knuth Challenge and Contributions

TAOCP: Compute $\lambda(Q_n)$ for some $n > 6$

No new chromatic number yet, but progress:

- Improved the upper bound for $n = 7$: $\lambda(Q_7) \leq 13$
- Improved the upper bound for $n = 8$: $\lambda(Q_8) \leq 14$
- Improved the lower bound for $n = 7$: $\lambda(Q_7) \geq 12$

n	1	2	3	4	5	6	7	8	9	10	11	12
upper bound	2	4	6	7	8	10	**13**	**14**	15	15	15	16
lower bound	2	4	6	7	8	10	**12**	*12*	12	13	14	15

Encoding

Each vertex has at least one color (using span s)

$$\bigvee_{c \in \{0,\ldots,s\}} x_{u,c}.$$

Two nodes u and v at distance 2 have a different color

$$\overline{x_{u,c}} \vee \overline{x_{v,c}}.$$

The colors of adjacent nodes u and v differs by at least 2

$$\overline{x_{u,c_1}} \vee \overline{x_{v,c_2}}.$$

with $|c_1 - c_2| \leq 1$

Symmetry Breaking

Type I: Fix node $\vec{0}$ to color c

- Note that color c might not appear
- What is the best color to fix?

$$x_{\vec{0},c} \vee \overline{x_{u,c}} \qquad \forall u \in \{0,1\}^n \setminus \{\vec{0}\}$$

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Type II: Sort the colors of the neighbors of node $\vec{0}$

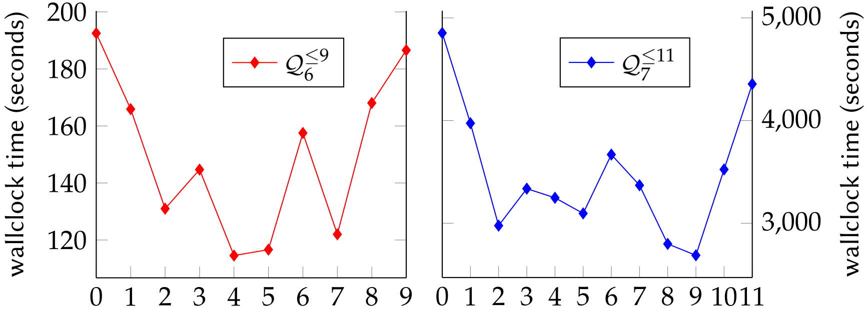
- All neighbors must have different colors
- All neighbors differ at least by 2 with $\vec{0}$

$$\overline{x_{\mathbf{e}_i,a}} \vee \overline{x_{\mathbf{e}_{i+1},b}} \qquad \forall i \in \{1..n-1\}, a \in \{0..s\}, b \in \{0..a-1\}$$

with $\mathbf{e}_i$ denoting the node with only bit i true.

Impact of Fixing Color of $\vec{0}$

Type I: Fix node $\vec{0}$ to color c



Observations:

- There is quite some noise since image not reflective
- Fixing color 2 (or its complement) works well in general

Impact of Symmetry Breaking

Evaluation on the existing bounds of Q_6 and Q_7

- Type I: Fix node $\vec{0}$ to color c
- Type II: Sort the colors of the neighbors of node $\vec{0}$
- All times in seconds

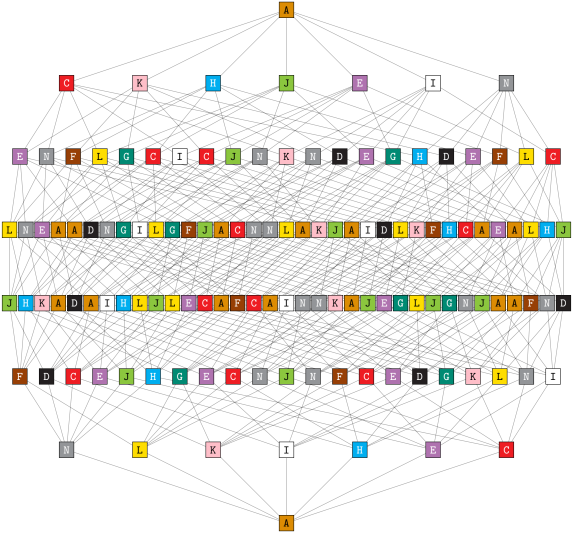
Instance	no SBPs	type I	type II	type I + type II
$\mathcal{Q}_6^{\leq 9}$	4679	102	0.87	0.05
$\mathcal{Q}_7^{\leq 10}$	$>$ 12 hrs	$>$ 12 hrs	183.81	7.64

Upper Bound: Local Search

For upper bound formulas, local search solvers are much stronger compared to CDCL

- We used `PalSAT` on a 128 core machine
- The formula for $\lambda(Q_7) \leq 13$ was solved in 11.67 (s)
- The formula for $\lambda(Q_8) \leq 14$ was solved in 316.01 (s)
- Fixing an internal symmetry reduces the costs of the formula for $\lambda(Q_8) \leq 14$ to just 1.93 (s)

Improved Q_7 Coloring



A color 0 C color 2 D color 3 E color 4 F color 5 G color 6 H color 7
I color 8 J color 9 K color 10 L color 11 N color 13

Missing Colors of Improved Q_7 Coloring

Color 0	0, 7, 25, 30, 42, 45, 51, 52, 75, 76, 82, 85, 97, 102, 120, 127
Color 2	6, 21, 26, 33, 47, 60, 67, 72, 95, 100, 114, 121
Color 3	11, 12, 48, 55, 81, 86, 106, 109
Color 4	2, 20, 31, 37, 46, 57, 71, 73, 90, 96, 115, 124
Color 5	13, 19, 40, 54, 78, 80, 107, 117
Color 6	1, 24, 34, 59, 68, 93, 103, 126
Color 7	15, 22, 44, 53, 74, 83, 105, 112
Color 8	3, 29, 38, 56, 64, 94, 101, 123
Color 9	4, 10, 17, 41, 50, 63, 77, 87, 88, 99, 110, 116
Color 10	27, 28, 32, 39, 65, 70, 122, 125
Color 11	5, 8, 18, 43, 49, 62, 79, 84, 89, 98, 108, 119
Color 13	9, 14, 16, 23, 35, 36, 58, 61, 66, 69, 91, 92, 104, 111, 113, 118

Some patterns:

- No node has color 1 or 12
- For $c \in \{0, 6, 7, 13\}$, if i has color c then $127 - i$ has color c

Internal Symmetry of Improved Q_7 Coloring

Color 0	5, 8, 30, 38, 43, 49, 67, 84, 89, 108, 114, 127
Color 1	16, 27, 55, 60, 70, 77, 97, 106
Color 2	1, 12, 22, 34, 47, 57, 75, 85, 88, 100, 115, 126
Color 3	19, 29, 52, 58, 64, 78, 103, 105
Color 4	7, 10, 32, 45, 86, 91, 113, 124
Color 5	9, 21, 35, 63, 79, 80, 101, 122
Color 6	4, 24, 46, 50, 66, 93, 104, 119
Color 7	3, 31, 41, 53, 69, 90, 111, 112
Color 8	14, 18, 36, 56, 72, 87, 98, 125
Color 9	0, 13, 39, 42, 81, 92, 118, 123
Color 10	20, 26, 51, 61, 71, 73, 96, 110
Color 11	6, 11, 17, 37, 40, 62, 76, 82, 95, 99, 116, 121
Color 12	23, 28, 48, 59, 65, 74, 102, 109
Color 13	2, 15, 25, 33, 44, 54, 68, 83, 94, 107, 117, 120

Internal symmetry, or non-trivial mapping onto itself:

- Apply the XOR mask 0000111
- Replace color c by $13 - c$

Internal Symmetry of Improved Q_8 Coloring

Color 0	2, 31, 40, 52, 68, 89, 111, 115, 140, 144, 167, 186, 203, 215, 225, 252
Color 1	14, 21, 24, 43, 67, 77, 86, 112, 147, 160, 173, 182, 200, 238, 245, 251
Color 2	9, 18, 36, 49, 62, 64, 91, 103, 106, 125, 133, 138, 159, 163, 184, 198, 209, 220, 233, 242
Color 3	7, 28, 34, 45, 59, 78, 85, 97, 118, 120, 128, 150, 153, 174, 181, 195, 205, 218, 228, 255
Color 4	4, 10, 17, 55, 73, 82, 95, 108, 143, 169, 178, 188, 212, 231, 234, 241
Color 5	13, 22, 33, 46, 56, 71, 92, 98, 117, 123, 131, 149, 154, 164, 191, 192, 206, 217, 237, 246
Color 6	0, 27, 39, 50, 61, 74, 81, 100, 105, 126, 134, 137, 156, 170, 177, 197, 210, 223, 227, 248
Color 8	1, 26, 38, 51, 60, 75, 80, 101, 104, 127, 135, 136, 157, 171, 176, 196, 211, 222, 226, 249
Color 9	12, 23, 32, 47, 57, 70, 93, 99, 116, 122, 130, 148, 155, 165, 190, 193, 207, 216, 236, 247
Color 10	5, 11, 16, 54, 72, 83, 94, 109, 142, 168, 179, 189, 213, 230, 235, 240
Color 11	6, 29, 35, 44, 58, 79, 84, 96, 119, 121, 129, 151, 152, 175, 180, 194, 204, 219, 229, 254
Color 12	8, 19, 37, 48, 63, 65, 90, 102, 107, 124, 132, 139, 158, 162, 185, 199, 208, 221, 232, 243
Color 13	15, 20, 25, 42, 66, 76, 87, 113, 146, 161, 172, 183, 201, 239, 244, 250
Color 14	3, 30, 41, 53, 69, 88, 110, 114, 141, 145, 166, 187, 202, 214, 224, 253

Internal symmetry, or non-trivial mapping onto itself:

- Apply the XOR mask 00000001
- Replace color c by $14 - c$

Lower Bound: Guiding the Solver

Fact

For $n \geq 2$, $\lambda(Q_n) \geq n + 2$.

Informal proof:

- Some node u has not the min or max color
- Node u blocks 3 colors
- The n neighbors of u block n additional colors

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Informal proof:

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SAT solver doesn't see it directly. Add the clauses:

$$\left(\bigvee_{c=0}^{m-1} x_{\mathbf{e}_i,c} \right) \vee \overline{x_{\mathbf{e}_{i+1},m}} \qquad \forall i \in \{1..n-1\}, \forall m \in \{0..s\}$$

Lower Bound: Cube and Conquer

Construct a tautological DNF ψ

$$\mathrm{SAT}(\Gamma) \iff \mathrm{SAT}(\Gamma \wedge \psi) \iff \mathrm{SAT}\left(\bigvee_{i=1}^{m} (\Gamma \wedge c_i)\right) \iff \mathrm{SAT}\left(\bigvee_{i=1}^{m} \Gamma_i\right)$$

All Γ_i formulas can be solved in parallel

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All Γ_i formulas can be solved in parallel

Automated split by march_cc is not effective

- Subformulas are solved on a machine with 128 cores

	depth			
	15	16	17	18
$\mathcal{Q}_7^{\leq 10}$	0.32	0.17	0.16	0.18
$\mathcal{Q}_7^{\leq 11}$	$>$ 16 hrs	$>$ 16 hrs	$>$ 16 hrs	$>$ 16 hrs

Lower Bound: Custom Split

Pick a set of nodes S and assign them to all possible values:

- $S = \{0011, 0101, 0101, 1001, 1010, 1100\}$ + prefix 000
- None of them can have the color of node $\vec{0}$
- The symmetry breaking fixes many neighbors of S

	custom split	no split
$\mathcal{Q}_7^{\leq 10}$	0.8	7.64
$\mathcal{Q}_7^{\leq 11}$	2588	$>$ 48 hrs

Conclusions

Knuth's Challenge: Compute $\lambda(Q_n)$ for some $n > 6$

- Improved the upper bound for $n = 7$: $\lambda(Q_7) \leq 13$
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Optimal/best-known colorings have internal symmetries

- How to exploit internal symmetries for upper bounds?

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Optimal/best-known colorings have internal symmetries

- How to exploit internal symmetries for upper bounds?

Symmetry breaking + a custom split crucial for $\lambda(Q_7) \geq 12$

- Which tricks can be used to reduce the computation?

Challenges

1. Prove or disprove that $\lambda(Q_n) \leq 2n - 1$ for every $n \geq 4$
2. Prove or disprove that $\lambda(Q_n) \geq n + 4$ for every $n \geq 6$
   - Note that this would solve Knuth's challenge: $\lambda(Q_{11}) \leq 15$
3. Develop a local search algorithm that can find the currently known upper bounds for $\lambda(Q_n)$ for $n \leq 12$
4. Develop effective set of clauses that guide solvers to quickly obtain proofs of $\lambda(Q_n) \geq n + 3$ for $n \leq 15$

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Thanks!

Questions

- Do you expect that $\lambda(Q_7) = 12$ or $\lambda(Q_7) = 13$?
- Which hypercube do you expect is easiest to solve?
- Do you have ideas to generate a good split automatically?

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